Local Perturbation Analysis of Linear Programming with Functional Relation Among Parameters

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ABSTRACT

In this paper, the authors study the sensitivity analysis for a class of linear programming (LP) problems with a functional relation among the objective function parameters or those of the right-hand side (RHS). The classical methods and standard sensitivity analysis software packages fail to function when a functional relation among the LP parameters prevail. In order to overcome this deficiency, the authors derive a series of sensitivity analysis formulae and devise corresponding algorithms for different groups of homogenous LP parameters. The validity of the derived formulae and devised algorithms is corroborated by open literature examples having linear as well as nonlinear functional relations between their vector b or vector c components.

Keywords: Dependent Parameters, Functional Relation, Linear Programming, Perturbation Analysis, Sensitivity Analysis

INTRODUCTION

The main purpose of classical sensitivity analysis is to examine the variations of the objective function’s optimal values and the solution components as a result of the infinitesimal changes in one of the parameters while the other ones are kept fixed (Bradley et al., 1977). One of the problems confronted in classical sensitivity analysis is when several parameters are changed simultaneously. In those cases, classical methods cannot obtain the effect of the perturbations on the objective function as well as on the optimal solution because when the simultaneous changes happen in the amplitudes range of basic variables or binding constraints, the order of basic solution changes. However, there is a conservative bound for the objective function coefficients or the right-hand side (RHS) parameters on their simultaneous changes. This bound is introduced under the 100 percent rule of changes when the perturbed
coefficients of objective function are related to basic variables, or when the RHS parameters are related to binding constraints. The 100 percent rule states that if the sum of the proper fraction of the desired changes to the maximum possible change in that direction is less than or equal to one then the current basic optimal solution will remain unchanged (Bradley et al., 1977).

The general form of linear programming (LP) problem can be considered as follows:

\[
\max_x \{ c^T x \mid A x \leq b, \quad x \geq 0 \}
\]

where \( A \) is an \( m \times n \) matrix with full rank and \( b \) is a column vector called the RHS parameters resembling the amount of resources, and \( c \) is the coefficient vector of the objective function; they are collectively called the parameters of the LP problem. A simplex algorithm may be used to solve the LP problems. A simplex algorithm, in each step, chooses a set of the independent columns from matrix \( A \), provides the correspondent basic feasible solutions and checks for the optimality conditions. It is assumed that a basic optimal solution is available for this problem.

Let us introduce the following mathematical notations and definitions used throughout this paper and for the simplex table:

- \( S_B = \{ B_1, B_2, \ldots, B_m \} \): The set of basic variable indices
- \( S_N = \{ N_1, N_2, \ldots, N_{n-m} \} \): The set of non-basic variable indices
- B: A sub-matrix of \( A \) whose columns are associated with the basic variable of \( S_B \)
- N: A sub-matrix of \( A \) whose columns are associated with the non-basic variable of \( S_N \)
- \( c_B^* \): The coefficient vector of the objective function whose elements are related to the basic variable
- \( x_B^* = B^{-1} b \): The optimal basic solution of the LP problem
- \( z^* = c_B^T \ b = B^{-1} \ b \): The optimal value of the objective function

\( Y = B^{-1} N \): The matrix with entries of the final table of the simplex for the non-basic variables

\( y_{ij} \): The entries of matrix \( Y \)

In modeling and solving LP problems, some cases occur where the parameters of the problem are functionally related in a way that the change of a specific parameter causes the simultaneous change of others. What separates the discussion of the sensitivity analysis in this paper from its classical methods is the functional relation among the parameters in LP. If the LP problem is in the form of (1), the functional relation among the parameters is defined as \( G(\theta) = 0 \), where the functional relation is considered among components of homogenous parameters of \( \theta \), (e.g., \( G(b_1, b_2, \ldots, b_m) = 0 \)). This relation can be linear or nonlinear which herein is considered to be continuous, differentiable, and its range space is one dimensional. The domain of \( G(\theta) = 0 \) is defined on an \( \varepsilon \) -neighborhood of \( \theta^0 \), namely, \( N_\varepsilon(\theta^0) = \{ \theta : \|\theta - \theta^0\| \leq \varepsilon \} \), where \( \theta^0 \) is the initial estimation of the perturbed parameters and it is not empty. Two situations of functional relation between problem parameters are cited below:

a. The nature of particular problems or solving methods imposes functional relation on LP problem. For example, the transportation simplex is developed with the prevalence of balance between supply and demand (namely the RHS parameters of transportation problem (Wendell, 1981)).

b. Because of the environmental and physical conditions inherent in the problems under the study, the functional relation often becomes necessary in the process of modeling (e.g., when the value of objective function coefficients computed by the analytical hierarchy process (AHP); then there is a linear function among the coefficients (Ghodsypour & O’Brien, 1998).
The rest of the paper is organized as follows. We present a review of the relevant literature on sensitivity analysis in the next section. We follow this review with a discussion of the sensitivity analysis for the LP problems with functional relations between the homogeneous parameters. We then provide four numerical examples illustrating the efficiency of the proposed methods in comparison to the classical methods. Finally, we conclude with our conclusions and future research directions.

THE LITERATURE REVIEW

The importance of sensitivity analysis in linear programming has been widely stressed in the management science literature. However, the research on sensitivity analysis with functional relation has been sporadic and scattered. Usually variation occurs in the RHS of the constraints and/or the objective function coefficients. A comprehensive survey of sensitivity analysis of LP problems can be found in Gal and Greenberg (1997) and Roos et al. (1997) and the references therein, but no-simultaneous perturbation of both the RHS and the objective function data is considered. Greenberg (2000) studied simultaneous perturbation of the RHS and the objective function data when the primal and dual linear optimization problems are in canonical form. However, for linear optimization problems in canonical form it is necessary to define the optimal partition to separate the active and inactive variables as well as the active and inactive constraints for all optimal solutions. In this section, we briefly review the state of the art in local perturbation (sensitivity) analysis. This review is confined to parametric analysis, range analysis and tolerance approach.

The post optimality analysis of the LP problems was introduced by Arsham (1992). This was the first effort to study sensitivity analysis on a specific structure of LP problems with functional relation among its parameters. This outcome stemmed from the fact that the balance between supply and demand imposed a functional relation on the RHS parameters of the transportation problem (Arsham, 1992). Arsham and Kahan (1989) used the required information for conducting sensitivity analysis from the transportation simplex. Ghodsypour and O’Brien (1998) introduced a method for supplier selection and optimal purchase by integrating AHP with LP where the objective function coefficient values were obtained on the basis of managers’ expert judgment in the process of using AHP. The objective function coefficients of the supplier selection problem followed a linear functional relation. Prevalence of this relationship prevented using the common methods of sensitivity analysis for the said problem. Ghodsypour and O’Brien (1998) studied this shortcoming and called for the need to develop conventional methods for sensitivity analysis when functional relations prevail.

Jansen et al. (1997) warned researchers and practicing managers against the confusing results when degeneracy of the optimal basic solution is ignored and proved that the break points in the optimal objective value (as a function of b and c) occur where the optimal partition as well as the representation of the optimal function changes. The intervals between two consequent break points are the unions of the ranges obtained from individual optimal bases by a simplex method with the common property of identical representation for the optimal value function, a fact which was previously proven by Adler and Monteiro (1992).

Koltai and Terlaky (2000) introduced three points of view with respect to sensitivity analysis of LP problems. The first point defined a space for the changes of the problem’s parameters, in a way that the optimal basis remained optimal. The second point defined a space for the parameter variations where the indices set of the optimal solution (not necessarily basic variables) did not change. The third point determined the bounds for the parameter changes where representation of the optimal value function (in uniparametric case, its slope) remained unchanged. These differences were all due to the degeneracy of the problem. For the multi-parametric case, after revising the tolerance approach, Filippi (2005) obtained the maximum
permitted variation of the parameters using a geometrical method. He found unique ranges which determined the maximum and minimum simultaneous changes for independent parameters. Ghaffari-Hadigheh and Terlaky (2006a) further analyzed the properties of the optimal value function when simultaneous perturbation of the RHS and the objective function happen in convex quadratic optimization.

Recently, sensitivity analysis in bi-parametric linear optimization problems has been the focus of several studies (Ghaffari-Hadigheh & Terlaky, 2006a, 2006b; Ghaffari-Hadigheh et al., 2006; Ghaffari-Hadigheh et al., 2007; Dehghan et al., 2007; Mirnia & Ghaffari-Hadigheh, 2007). Ghaffari-Hadigheh et al. (2008) have proved that invariancy regions are separated by vertical and horizontal lines and generate a mesh-like area. They have also showed that the boundaries of these regions can be identified in polynomial time.

**SENSITIVITY ANALYSIS WITH FUNCTIONAL RELATION**

In this section, we focus on local perturbation (sensitivity) analysis and not the parametric one which is well explained in Fiacco (1983). As a result, the range of variation of parameters is very small and falls within an $\varepsilon$-neighborhood of the estimated parameters. We devise sensitivity analysis methodologies for problems with functional relation among the “homogenous” parameters. This functional relation can be linear or nonlinear for the RHS parameter values or for the objective function coefficient values. We should reiterate that when the functional relation is a nonlinear function, very small perturbation for parameters values are considered. It is assumed that variations fall within an $\varepsilon$-neighborhood of the estimated parameters. It is also assumed that a basic optimal solution is available. However, if the underlying optimal basic solution is degenerate, then, the method might fail to give a proper result (See Jansen et al., 1997). Moreover, the method is not applicable if the optimal solution is not a basic one.

**Variations in the RHS Values**

In this section, first the slope changes of the optimal values of the objective function and the basic variables are analyzed with respect to the changes in RHS parameter values regardless of the prevalence of a functional relation. Then, using the total differentiation operator, the method for calculating the slope changes in case functional relations among the RHS parameters are provided. A flowchart of the sensitivity analysis with respect to the changes in the RHS values is presented in Figure 1.

Let us consider solving problem (1) on the basis of the primary estimation of $\mathbf{b}$ and obtain the optimal solution:

$$z^* = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b}$$

$$x^*_B = \mathbf{B}^{-1} \mathbf{b}$$

When there is no relationship between the components of vector $\mathbf{b}$, then we have:

$$\frac{dz^*}{db_i} = \mathbf{c}_B^T \mathbf{B}^{-1} e_i \quad \text{or} \quad \frac{dz^*}{db_i} = v_i \quad i = 1, 2, \ldots, m$$

$$\frac{dx^*_j}{db_k} = y_{i,k}^* \quad k = 1, 2, \ldots, m$$

where $v_i$ is the value of dual variable corresponding to the shadow price of the $i^{th}$ constraint. Also,

$$G(b_1, \ldots, b_m) = 0$$
Then, the classical methods of the sensitivity analysis addressed above will not yield the correct results. Under such functional prevalence, one may use total differentiation as follows:

\[
\frac{dz^*}{b_1} db_1 + \frac{dz^*}{b_2} db_2 + \ldots + \frac{dz^*}{b_m} db_m = dz^*
\]  

(7)

\[
\frac{dx_{B_i}}{b_1} db_1 + \frac{dx_{B_i}}{b_2} db_2 + \ldots + \frac{dx_{B_i}}{b_m} db_m = dx_{B_i}^*
\]  

(8)

Then, the classical methods of the sensitivity analysis addressed above will not yield the correct results. Under such functional prevalence, one may use total differentiation as follows:

\[
\frac{\partial G}{\partial b_1} db_1 + \frac{\partial G}{\partial b_2} db_2 + \ldots + \frac{\partial G}{\partial b_m} db_m = 0
\]  

(9)

Herein, sensitivity analysis of \( z^* \) and \( x_{B_i}^* \) with respect to \( b_i \)'s will be derived for the parameter ranges where the basic solution remains optimal. In this case, we consider the following two forms:

- Only two components of vector \( b \) are perturbed simultaneously.
- \( k \) elements of vector \( b \) are perturbed simultaneously (\( 2 \leq k \leq m \)).

Figure 1. Flowchart of the sensitivity analysis with respect to the changes in the RHS values
The sensitivity analyses for both of the above cases are discussed below:

**Perturbation of Two RHS Parameters**

Suppose that the effect of perturbation $b_1$ on $z^*$ and $x_{B1}^*$ is studied in a way that this perturbation is absorbed by a second parameter, $b_m$, so that the prevalent functional relation is satisfied. In such a case, we will have:

$$\frac{\partial G}{\partial b_i} db_i + \frac{\partial G}{\partial b_m} db_m = 0$$

$$db_m = - \frac{\partial G / \partial b_i}{\partial G / \partial b_m} db_i$$

(10)

Also, from (7):

$$dz = \frac{\partial z^*}{\partial b_i} db_i + \frac{\partial z^*}{\partial b_m} db_m$$

$$dz = \left[ \frac{\partial z^*}{\partial b_i} + \frac{\partial z^*}{\partial b_m} \left( - \frac{\partial G / \partial b_i}{\partial G / \partial b_m} \right) \right] db_i$$

(11)

Similarly, from (8), we have:

$$dx_{B1}^* = \left[ \frac{\partial x_{B1}^*}{\partial b_i} + \frac{\partial x_{B1}^*}{\partial b_m} \left( - \frac{\partial G / \partial b_i}{\partial G / \partial b_m} \right) \right] db_i$$

(12)

Then, we obtain the maximum permitted changes of parameter $b_1$ in order that the current optimal solution remains feasible. Using (11) and (12), the obvious effect of parameter $b_1$ can be respectively estimated on $z^*$ and $x_{B1}^*$, as follows: (If function $G$ is linear, the effects of changes are calculated accurately).

$$\Delta z^* = v_1 + v_m \left( - \frac{\partial G / \partial b_1}{\partial G / \partial b_m} \right) \Delta b_1$$

(13)

$$\Delta x_{B1}^* = y_{i,1} + y_{i,m} \left( - \frac{\partial G / \partial b_1}{\partial G / \partial b_m} \right) \Delta b_1$$

(14)

These two relations are valid when $G$ is a linear function. When $G$ is a nonlinear one, these relations may be valid for very small perturbation of parameters. As the deviation gets larger, the linear approximation of the nonlinear function $G$ will no longer be satisfied.

The optimality solution point under simultaneous perturbation of the RHS parameters is studied in two cases:

**A. General variations on RHS Parameters**

In this case, it is sufficient to keep the feasibility of the solution as:

$$x_{B1}^* + \Delta x_{B1}^* \geq 0 \quad i = 1, 2, \ldots, m$$

$$x_{B1}^* + \left[ y_{i,1} + y_{i,m} \left( - \frac{\partial G / \partial b_1}{\partial G / \partial b_m} \right) \right] \Delta b_i \geq 0$$

Therefore,

$$x_{B1}^* + \frac{dx_{B1}^*}{db_1} \Delta b_1 \geq 0$$

If $\frac{dx_{B1}^*}{db_1} < 0$, then
\( \Delta b_1 \leq \frac{-x_{R_i}^*}{dx_{R_i}^* / db_1} \)

and in the general case, we have:

\[
\Delta b_1 \leq \min_{\bar{R}_i \in \mathcal{S}_n} \left\{ -\frac{x_{R_i}^*}{dx_{R_i}^* / db_1} \left| \frac{dx_{R_i}^*}{db_1} < 0 \right. \right\}
\]

(15)

**B. Variations in RHS Parameters Related to the Binding Constraints**

In this case, the basic solution will remain unchanged, provided that the 100 percent rule of variations is granted (Bradley et al., 1977), that is:

\[
\frac{\Delta b_1}{\Delta b_1^{\text{max}}} + \frac{\Delta b_m}{\Delta b_m^{\text{max}}} \leq 1
\]

(16)

and \( \Delta b \) and \( \Delta b_m^{\text{max}} \) must have the same sign and be finite,

\[
\frac{\Delta b_1}{\Delta b_1^{\text{max}}} + \frac{\Delta b_m^{\text{max}}}{\Delta b_m^{\text{max}}} \leq 1
\]

\[
\Delta b_1 \left[ \frac{1}{\Delta b_1^{\text{max}}} + \frac{-\partial G / \partial b_1}{\Delta b_m^{\text{max}}} \right] \leq 1
\]

and the upper and lower bounds are defined as follows according to the sign of \( \Delta b_1 \):

**B.1) if \( \Delta b_1 > 0 \)**

\[
\Delta b_1 \leq \frac{1}{\Delta b_1^{\text{max}}} + \frac{-\partial G / \partial b_1}{\Delta b_m^{\text{max}}}
\]

(17)

Then, \( \Delta b_1^{\text{max}} \) is the maximum increase of \( b_1 \) from its initial value and \( \Delta b_m^{\text{max}} \) is related to the two following cases:

**B.1.1) If \( \frac{\partial G / \partial b_1}{\partial G / \partial b_m} \geq 0 \), \( \Delta b_1 \) is non-positive and \( \Delta b_m^{\text{max}} \) is the maximum decrease of \( b_m \) from its initial value.**

**B.1.2) If \( \frac{\partial G / \partial b_1}{\partial G / \partial b_m} < 0 \), \( \Delta b_1 \) is positive and \( \Delta b_m^{\text{max}} \) is the maximum increase of \( b_m \) from its initial value.**

**B.2) If \( \Delta b_1 < 0 \)**

\[
\Delta b_1 \geq \frac{1}{\Delta b_1^{\text{max}}} + \frac{-\partial G / \partial b_1}{\Delta b_m^{\text{max}}}
\]

(18)

Then, \( \Delta b_m^{\text{max}} \) is the maximum decrease of \( b_m \) from its initial value and \( \Delta b_m^{\text{max}} \) is related to the two following cases:

**B.2.1) If \( \frac{\partial G / \partial b_1}{\partial G / \partial b_m} \geq 0 \), \( \Delta b_m \) is non-negative and \( \Delta b_m^{\text{max}} \) is the maximum increase of \( b_m \) from its initial value.**

**B.2.2) If \( \frac{\partial G / \partial b_1}{\partial G / \partial b_m} < 0 \), \( \Delta b_m \) is negative and \( \Delta b_m^{\text{max}} \) is the maximum decrease of \( b_m \) from its initial value.**
Perturbation of \( k \) Components of Vector \( b \) (General Case)

The sensitivity analysis of \( z^* \) and \( x_{B_i}^* \) should be studied when \( k \) components of vector \( b \) change and others remain fixed. This happens when one wants to study the effect of the change of \( b_i \) on \( z^* \) and \( x_{B_i}^* \) so that the other \((k-1)\) parameters change. In this case, the variations of \((k-1)\) parameters \( \Delta b_i \) are arbitrarily changed and \( k^{th} \) parameter is obtained from functional relation \( G \).

\[
\Delta z^* = \frac{dz^*}{db_i} \Delta b_i = \left[ \frac{\partial z^*}{\partial b_i} + \frac{\partial z^*}{\partial b_i} \frac{db_2}{db_i} + \ldots + \frac{\partial z^*}{\partial b_k} \frac{db_k}{db_i} \right] \Delta b_i
\]

(19)

Here, for simplicity, the first \( k \) parameters are taken for the above analysis without loss of generality. Similarly, we have:

\[
\Delta x_{B_i}^* = \frac{dx_{B_i}^*}{db_i} \Delta b_i = \left[ \frac{\partial x_{B_i}^*}{\partial b_i} + \frac{\partial x_{B_i}^*}{\partial b_i} \frac{db_2}{db_i} + \ldots + \frac{\partial x_{B_i}^*}{\partial b_k} \frac{db_k}{db_i} \right] \Delta b_i
\]

(20)

Now, by replacing \( \frac{dz^*}{db_i} = v_i \) and \( \frac{dx_{B_i}^*}{db_i} = y_{i,k}^* \)，we have:

\[
\Delta z^* = \left[ v_1 + v_2 \frac{\Delta b_2}{\Delta b_1} + \ldots + v_k \frac{\Delta b_k}{\Delta b_1} \right] \Delta b_1
\]

\[
\Delta x_{B_i}^* = \left[ y_{i,1}^* + y_{i,2}^* \frac{\Delta b_2}{\Delta b_1} + \ldots + y_{i,k}^* \frac{\Delta b_k}{\Delta b_1} \right] \Delta b_1
\]

In above equation, we replace \( \frac{db_k}{db_i} \) with \( \frac{\Delta b_k}{\Delta b_i} \). The feasibility of the solution due to the simultaneous changes of the RHS parameters is studied in two cases:

**A) General Variations on the RHS Parameters**

In this case, in order to remain the solution feasible, it is sufficient that:

\[
x_{B_i}^* + \Delta x_{B_i}^* \geq 0 \quad i = 1, 2, \ldots, m
\]

Therefore,

\[
x_{B_i}^* + \frac{dx_{B_i}^*}{db_i} \Delta b_i \geq 0
\]

If \( \frac{dx_{B_i}^*}{db_i} < 0 \), then

\[
\Delta b_i \leq \frac{-x_{B_i}^*}{\frac{dx_{B_i}^*}{db_i}} / \frac{db_i}{db_i}
\]

and in general, we have

\[
\Delta b_i \leq \min_{n \in S_n} \left\{ \frac{-x_{B_i}^*}{\frac{dx_{B_i}^*}{db_i}} \left| \frac{dx_{B_i}^*}{db_i} < 0 \right. \right\}
\]

(21)

**B) Variations of the RHS Parameters Related to the Binding Constraints**

In this case, the basic solution remains optimal, provided that the 100 percent rule of changes is satisfied (Bradley et al., 1977), that is:

\[
\sum_{i=1}^{k} \frac{\Delta b_i}{\Delta b_{i}^{max}} \leq 1
\]

where \( \Delta b_i \) and \( \Delta b_{i}^{max} \) have the same sign and are finite.
Variations in the Objective Function Coefficients

An indirect approach for analyzing the sensitivity values with respect to the coefficients of the objective function is to use the dual problem because the objective function coefficients in the primal problem play the role of the RHS parameters in the dual one.

Here, a direct approach is used to do the sensitivity analysis with respect to the perturbations in the objective function coefficients. The sequence of the steps necessary for conducting the sensitivity analysis with respect to perturbations of two objective function coefficients is shown in Figure 2.

Let us consider solving problem (1) on the basis of the initial value of the component of vector $c$ and obtain the optimal solution:

$$z^* = c_B^T B^{-1} b$$

$$x_B^* = B^{-1} b$$

$$z_j - c_j = c_B y_j - c_j = \sum c_{B_i} y_{i,j} - c_j$$

If there is no functional relation among the components of $c$, we have:

$$\frac{dz^*}{dc_k} =$$

- $x_{B_i}^*$ if $c_k = c_{B_i}$
- 0 if $c_k$ corresponds to nonbasic variables (22)

Figure 2. Flowchart of the sensitivity analysis with respect to the changes in the objective function coefficient values
\[ \frac{d(z_j - c_j)}{dc_k} = \begin{cases} 
0 & \text{if } k \text{ corresponds to nonbasic variables and } k \neq j \\
y_{k,j} & \text{if } k \text{ corresponds to basic variables and } k \neq j \\
0 & \text{if } k \text{ corresponds to basic variables and } k = j \\
-1 & \text{if } k \text{ corresponds to nonbasic variables and } k = j 
\end{cases} \]  

(23)

Equations (22) and (23) respectively show the sensitivities of the objective function and the shadow price of each variable with respect to the perturbation of \( c_k \). In cases with functional relation among the components of vector \( c \):

\[ H(c_1, c_2, \ldots, c_n) = 0 \]  

(24)

Similar to the previous section, using the concept of total differentiation, we have:

\[ dz^* = \frac{\partial z^*}{\partial c_1} \, dc_1 + \frac{\partial z^*}{\partial c_2} \, dc_2 + \cdots \frac{\partial z^*}{\partial c_n} \, dc_n \]  

(25)

\[ d\left[(z_j - c_j)\right] = \frac{\partial (z_j - c_j)}{\partial c_1} \, dc_1 \\
+ \frac{\partial (z_j - c_j)}{\partial c_2} \, dc_2 + \cdots + \frac{\partial (z_j - c_j)}{\partial c_n} \, dc_n \]  

(26)

and also:

\[ \frac{\partial H}{\partial c_1} \, dc_1 + \frac{\partial H}{\partial c_2} \, dc_2 + \cdots + \frac{\partial H}{\partial c_n} \, dc_n = 0 \]  

(27)

The sensitivity analyses for both of the above cases are discussed below:

**Perturbation of Only Two Coefficients of the Objective Function**

Suppose that one wants to study the effect of perturbation of \( c_j \) on \( z^* \) and \( z_j - c_j \) such that another component of vector \( e \) like \( c_k \) is also perturbed and the functional relation between components of \( e \) is satisfied. In this case we have:

\[ \frac{\partial H}{\partial c_j} \, dc_j + \frac{\partial H}{\partial c_k} \, dc_k = 0 \]

\[ dc_k = -\frac{\partial H}{\partial H/\partial c_k} \, dc_j \]  

(28)

and also:

\[ dz^* = \frac{\partial z^*}{\partial c_j} \, dc_j + \frac{\partial z^*}{\partial c_k} \, dc_k = \left[ \frac{\partial z^*}{\partial c_j} \right] \frac{\partial H}{\partial c_j} \, dc_j \]  

(29)

where equation (29) shows the sensitivity of \( z^* \) with respect to the perturbation of \( c_j \). These two relations are valid when \( H \) is a linear function. If \( H \) is a nonlinear one, these relations may be valid for very small perturbation of parameters. As the deviations increase in size, linear approximation of the nonlinear function \( H \) is no longer satisfied.

For replacing the equation (22) with (29), we consider the following states:

A) If both parameters are related to non-basic variables.

B) If one of the parameters is related to a basic variable and the other one corresponds to a non-basic one.

C) If both parameters are related to basic variables.
For each state, we calculate equation (29) as follows:

A) \( dz^* = 0 \)

1- If \( c_f \) relates to basic variables and \( c_k \) to non-basic variables then,

\[
dz^* = \left[ x_{b_j}^* + 0 \right] dc_j
\]

2- If \( c_f \) relates to basic variables and \( c_k \) to non-basic variables then,

\[
dz^* = \left[ 0 + x_{b_j}^* \left( -\frac{\partial H}{\partial H/\partial c_k} \right) \right] dc_j
\]

C) \( dz^* = \left[ x_{b_j}^* + x_{b_k}^* \left( -\frac{\partial H}{\partial H/\partial c_k} \right) \right] dc_j \)

Similarly, from (26):

\[
d\left( z_j - c_j \right) = \left[ \frac{\partial(z_j - c_j)}{\partial c_j} + \frac{\partial(z_j - c_j)}{\partial c_k} \left( -\frac{\partial H}{\partial H/\partial c_k} \right) \right] dc_j
\]

(30)

As mentioned above, when \( H \) is a nonlinear function, (30) will be valid for very small perturbations of perturbed parameters. For replacing (24) with (30), we consider the following states:

A) If both parameters are related to non-basic variables and

1- \( k, f \neq j \)

2- At least one of them corresponds to \( j \)

B) If one of the parameters is related to a basic variable and the other one is associated with a non-basic one and

1- Non-basic variables do not correspond to \( j \)

2- Non-basic variables correspond to \( j \)

C) If both parameters are related to basic variables.

For each of the above states, we calculate equation (30) as follows:

A.1) \( d\left( z_j - c_j \right) = 0 \)

A.1.1) \( c_f = c_j \)

\[
d\left( z_j - c_j \right) = \left[ -1 + 0 \right] dc_j
\]

A.1.2) \( c_k = c_f \)

\[
d\left( z_j - c_j \right) = \left[ 0 + (-1) \left( -\frac{\partial H}{\partial H/\partial c_k} \right) \right] dc_j
\]

B.1) \( c_f \) refers to basic variables and \( c_k \) refers to non-basic variables

B.1.1) \( k \neq j \)

\[
d\left( z_j - c_j \right) = \left[ y_{f,j} + 0 \right] dc_f
\]

B.1.2) \( k = j \)

\[
d\left( z_j - c_j \right) = \left[ y_{f,j} + (-1) \left( -\frac{\partial H}{\partial H/\partial c_k} \right) \right] dc_f
\]

B.2) \( c_f \) refers to non-basic variables and \( c_k \) refers to basic variables

B.2.1) \( f \neq j \)

\[
d\left( z_j - c_j \right) = \left[ 0 + y_{k,j} \left( -\frac{\partial H}{\partial H/\partial c_k} \right) \right] dc_k
\]

B.2.2) \( f = j \)

\[
d\left( z_j - c_j \right) = \left[ -1 + y_{k,j} \left( -\frac{\partial H}{\partial H/\partial c_k} \right) \right] dc_k
\]

C.1) \( k, f \neq j \)

\[
d\left( z_j - c_j \right) = \left[ y_{f,j} + y_{k,j} \left( -\frac{\partial H}{\partial H/\partial c_k} \right) \right] dc_f
\]

C.2) \( k = j \)

\[
d\left( z_j - c_j \right) = \left[ y_{f,j} + 0 \right] dc_f
\]

C.3) \( f = j \)

\[
d\left( z_j - c_j \right) = \left[ 0 + y_{k,j} \left( -\frac{\partial H}{\partial H/\partial c_k} \right) \right] dc_k
\]

Next, we derive the upper and lower bounds for the value of the parameter \( c_f \) (simultaneous change of \( c_k \)) so that the current optimal solution remains optimal. Keeping optimality of the solution under simultaneous changes of the
objective function coefficients is studied in two cases as follows:

**A) General Variations in Objective Function Coefficients**

In this case, in order to keep the solution optimal, it is sufficient that:

\[ j = 1, 2, \ldots, n \quad z_j - c_j + \Delta(z_j - c_j) \geq 0 \]

Considering (30), we have:

\[ \Delta(z_j - c_j) = \frac{d(z_j - c_j)}{dc_j} \Delta c_j \]

Thus

\[ z_j - c_j + \frac{d(z_j - c_j)}{dc_j} \Delta c_j \geq 0 \]

Now, if \( \frac{d(z_j - c_j)}{dc_j} < 0 \), then

\[ \Delta c_j \leq \frac{-(z_j - c_j)}{d(z_j - c_j)} \frac{d(z_j - c_j)}{dc_j} \]

Generally, we have:

\[ \Delta c_j \leq \min_{j \in S_n} \left\{ \frac{-(z_j - c_j)}{d(z_j - c_j)} \frac{d(z_j - c_j)}{dc_j} < 0 \right\} \]

\[ \Delta c_j \leq \frac{1}{\frac{\partial H}{\partial c_j}} + \frac{1}{\frac{\partial H}{\partial c_k}} \Delta c_k^{\text{max}} \leq 1 \]

**B) Variations in Coefficients Related to Basic Variables**

If the 100 percent rule for variations is satisfied, the basic solution will remain optimal, that is:

\[ \Delta c_f + \frac{\Delta c_k}{\Delta c_k^{\text{max}}} \leq 1 \]

\[ \Delta c_f \left( \frac{1}{\Delta c_f^{\text{max}}} + \frac{\frac{\partial H}{\partial c_f}}{\frac{\partial H}{\partial c_k}} \right) \leq 1 \]

But upper and lower bounds are defined according to the sign of \( \Delta c_f \) as follows:

If \( \Delta c_f > 0 \)

\[ \Delta c_f \leq \frac{1}{\frac{\partial H}{\partial c_f}} + \frac{1}{\frac{\partial H}{\partial c_k}} \Delta c_k^{\text{max}} \leq 1 \]

where \( \Delta c_f^{\text{max}} \) is the maximum increase of \( c_f \) from its initial value and \( \Delta c_k^{\text{max}} \) refers to the two following cases:

A.1) If \( \frac{\partial H}{\partial c_f} \geq 0 \), \( \Delta c_k \) is non-positive and \( \Delta c_k^{\text{max}} \) is the maximum decrease of \( c_k \) from its initial value.

A.2) If \( \frac{\partial H}{\partial c_f} < 0 \), \( \Delta c_k \) is positive and \( \Delta c_k^{\text{max}} \) is the maximum increase of \( c_k \) from its initial value.

B) If \( \Delta c_f < 0 \)

---

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\[ \Delta c_f \geq \frac{1}{\Delta c_f^\text{max}} + \frac{1}{\Delta c_k^\text{max}} \]  

(33)

\( \Delta c_f^\text{max} \) is the maximum decrease of \( c_f \) from its initial value and \( \Delta c_k^\text{max} \) is related to the two following cases:

**B.1)** If \( \partial H / \partial c_f \geq 0 \), \( \Delta c_f \) is non-negative and \( \Delta c_k^\text{max} \) is the maximum permitted increase of \( c_k \) from its initial value.

**B.2)** If \( \partial H / \partial c_f < 0 \), \( \Delta c_f \) is negative and \( \Delta c_k^\text{max} \) is the maximum permitted decrease of \( c_k \) from its initial value.

**Variations in k Components of the Objective Function Coefficients**

Suppose that the effect of change of \( c_1 \) on \( z^* \) and \( z_j - c_j \) is to be studied when other \( (k-1) \) parameters change and the functional relation remains satisfied.

Similar to the previous section, using the concept of total differentiation, we have:

\[
\Delta z^* = \frac{df}{dc_1} \Delta c_1 = \left[ \frac{\partial z^*}{\partial c_1} + \frac{\partial z^*}{\partial c_2} \frac{dc_2}{dc_1} + \cdots + \frac{\partial z^*}{\partial c_k} \frac{dc_k}{dc_1} \right] \Delta c_1
\]

(34)

Here, without loss of generality, for the sake of simplicity the first \( k \) parameters are assumed as the subject parameters. Similarly, we have:

\[
\Delta (z_j - c_j) = \frac{d(z_j - c_j)}{dc_1} \Delta c_1
\]

\[
\Delta (z_j - c_j) = \left[ \frac{\partial (z_j - c_j)}{\partial c_1} + \frac{\partial (z_j - c_j)}{\partial c_2} \frac{dc_2}{dc_1} + \cdots + \frac{\partial (z_j - c_j)}{\partial c_k} \frac{dc_k}{dc_1} \right] \Delta c_1
\]

(35)

Equations (22) and (23) determine the values of \( \frac{\partial z^*}{\partial c_k} \) and \( \frac{\partial (z_j - c_j)}{\partial c_k} \) respectively in equations (34) and (35).

Here, because the specific values of \( \frac{\partial z^*}{\partial c_k} \) and \( \frac{\partial (z_j - c_j)}{\partial c_k} \) are different and dependent on their corresponding basic variables, we only recall this general equation (see the section on the perturbation of only two coefficients of the objective function).

Retaining the solution optimality in the situation of simultaneous changes of the objective function coefficients (change of \( k \) parameters) is studied in two following cases:

**A) General Variations in Objective Function's Coefficients**

In this case, the basic solution remains optimal, if

\[
\frac{d(z_j - c_j)}{dc_1} \Delta c_1 \geq 0
\]

\[
j = 1, 2, \ldots, n
\]

If \( \frac{d(z_j - c_j)}{dc_1} < 0 \)

\[
\Delta c_1 \leq \frac{- (z_j - c_j)}{d(z_j - c_j)} \frac{d(z_j - c_j)}{dc_1}
\]

In the general case, we have:

\[
\Delta c_1 \leq \min_{j \notin S_1} \left[ \frac{- (z_j - c_j)}{d(z_j - c_j)} \left| \frac{d(z_j - c_j)}{dc_1} \right| < 0 \right]
\]

(36)
B) Variations in Objective Function’s Coefficients Corresponding to Basic Variables

If the 100 percent rule for variations is satisfied, the basic solution will remain optimal, that is:

$$\sum_{i=1}^{k} \frac{\Delta c_{yi}}{\Delta c_{y_{\text{max}}}} \leq 1$$  \hspace{1cm} (37)

**NUMERICAL EXAMPLES**

In this section, we corroborate the validity of our proposed algorithms by applying the formulae derived in this study to four numerical examples and comparing our results with those obtained by Ghodsypour and O’Brien (1998).

**Example 1: Supplier Selection Problem**

In this example, we consider a supplier selection problem. Supplier selection is a multicriteria problem with both qualitative and quantitative factors. In order to select the best suppliers it is necessary to make a trade-off between these qualitative and quantitative factors some of which may conflict. Ghodsypour and O’Brien (1998) have devised a procedure that integrates AHP and LP and considers both qualitative and quantitative factors in choosing the best suppliers and placing the optimum order quantities among them such that the total value of purchasing is maximized. Their method calculates the attraction of every supplier by considering desirability and management criteria. The amount of purchase from each supplier is the decision variable in their LP model and the attraction rates are used as the coefficients of the objective function along with constraints such as budget, amount of demand and product quality. Ghodsypour and O’Brien (1998) suggest examining the sensitivity of the estimated parameters in their model because the attraction rate of each supplier is extracted on the basis of expert judgment. However, Ghodsypour and O’Brien (1998) ignore the functional relations among the coefficients of their objective function. This functional relation is one of the characteristics of AHP where the sum of the objective function coefficients must equal 1. The prevalence of this linear relationship in the model prevents one to use the classical methods for doing the sensitivity analysis. This shortcoming could be overcome by using the proposed methods in this paper. To clarify the above assertion, we present the following problem formulation proposed by Ghodsypour and O’Brien (1998):

Maximize  

Subject to:

$$z = 0.297x_1 + 0.252x_2 + 0.268x_3 + 0.183x_4$$

$$0.03x_1 + 0.05x_2 + 0.01x_3 + 0.06x_4 \leq 20$$

$$x_1 + x_2 + x_3 + x_4 = 1000$$

$$x_1 \leq 400$$

$$x_2 \leq 700$$

$$x_3 \leq 600$$

$$x_4 \leq 500$$

$$x_j \geq 0$$

$$j = 1, 2, 3, 4$$  \hspace{1cm} (38)

The final (optimal) solution tableau reported by Ghodsypour and O’Brien (1998) is presented in Table 1.

If the objective function coefficient related to \( j \) variable is denoted by \( c_j \), the subject functional relation may be defined as:

$$H(c_1, c_2, c_3, c_4) = c_1 + c_2 + c_3 + c_4 = 1$$

The sensitivity analysis results reported by Ghodsypour and O’Brien (1998) are presented in Table 2.

The changes in parameters of the objective function coefficients are studied using the optimal solution and the sensitivity analysis results are presented in Tables 1 and 2, respectively, regardless of the functional relations shown in Table 3 (output of Lindo®). In order to conduct sensitivity analysis, the steps indicated in the flowchart presented in Figure 2 are taken.
For the purpose of using all steps of the flowchart, first the variations of \(c_2\) and \(c_3\) are considered (in this case, \(c_3\) is related to a non-basic variable and step (5) is not used). Then, the variation of \(c_4\) and \(c_3\) is studied (in this case, both parameters are related to basic variables and step (4) is not used).

A) Variations of \(c_2\) and \(c_3\)

\[
\begin{align*}
\text{Step 1:} & \text{ determine the desired value of change in the first parameter: } \Delta c_2 = 0.005. \\
\text{Step 2:} & \text{ calculate the value of change in the second parameter using the function } H: \\
& \Delta c_3 + \Delta c_4 = 0 \quad \longrightarrow \quad \Delta c_3 = -0.005
\end{align*}
\]
Step 3: are the changed parameters related to the basic variables?

No, so go to step 4.

Step 4: does the change of first parameter satisfy (31)?

In order to answer this question, the slope of \( z_j - c_j \) is calculated with respect to the perturbation of \( c_2 \).

\[
\frac{d(z_j - c_j)}{dc_2} = \frac{\partial(z_j - c_j)}{\partial c_2} + \frac{\partial(z_j - c_j)}{\partial c_3} \left( -\frac{\partial H}{\partial c_2} \right) = \left( 0 - y_{3,j} \right) \quad j \neq 2 \quad (*)
\]

\[
\frac{d(z_j - c_j)}{dc_2} = \left( -1 - y_{3,2} \right) \quad (**)\]

\( (*) \) and \( (**) \) result in:

\[
\frac{d(z_2 - c_2)}{dc_2} = -1 - 1 = -2
\]

\[
\frac{d(z_4 - c_4)}{dc_2} = -1 \quad \frac{d(z_6 - c_6)}{dc_2} = -(1) = 1
\]

\[
\Delta c_2 \leq \min\{ -\frac{0.016}{-2}, -\frac{0.085}{-1} \} = 0.008
\]

0.005 < 0.008

This means that, applying these changes, the basic solution remains optimal and valid.

Step 6: calculate the changes of \( z^* \) and the \( z_j - c_j \) s.

According to \( (*) \) and \( (**) \):

\[
\Delta(z_2 - c_2) = -0.01 \quad \Delta z^* = \Delta(z_2 - c_2) = 0.006 > 0
\]

\[
\Delta(z_4 - c_4) = -0.005 \quad \Delta z^* = \Delta(z_4 - c_4) = 0.08 > 0
\]

\[
\Delta(z_6 - c_6) = 0.005 \quad \Delta z^* = \Delta(z_6 - c_6) = 0.034 > 0
\]

\[
dz^* = \left( \frac{\partial z^*}{\partial c_2} + \frac{\partial z^*}{\partial c_3} \left( -\frac{\partial H}{\partial c_2} \right) \right) dc_2 \]

\[
\Delta z^* = [0 + x_{b_1}(-1)]\Delta c_2 \]

\[
\Delta z^* = [-600](0.005) = -3 \quad z^* = 276.6
\]

The changes are applied for testing:

\[
z^* = c_1x_1 + (c_2 + \Delta c_2)x_2 + (c_3 + \Delta c_3)x_3 + c_4x_4
\]

\[
z^* = 0.297 \times 400 + 0 + 0.016 \times 400 + 0 = 267.6
\]

---

**Table 3. The allowable variation bounds of the objective function coefficients**

<table>
<thead>
<tr>
<th>Var.</th>
<th>OFC</th>
<th>INC</th>
<th>DEC</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>0.297</td>
<td>( \infty )</td>
<td>0.029</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0.252</td>
<td>0.016</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>0.268</td>
<td>0.029</td>
<td>0.016</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>0.183</td>
<td>0.085</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>
According to step 4, the maximum permitted value of $c_2$ that can be increased under the condition of keeping the basic solution unchanged and optimal is: $\Delta c_2 \leq 0.008$.

Now let us assume that $\Delta c_2 = 0.01$; then, the change of the basis should be considered. For this purpose, the values of $z_j - c_j$ in step 4 are updated using the resulted slopes.

$$
\Delta(z_2 - c_2) = -0.02 \quad \rightarrow \quad z_2 - c_2 = -0.004 < 0
$$

$$
\Delta(z_1 - c_1) = -0.01 \quad \rightarrow \quad z_1 - c_1 = 0.075 > 0
$$

$$
\Delta(z_6 - c_6) = 0.01 \quad \rightarrow \quad z_6 - c_6 = 0.039 > 0
$$

This information shows that $x_2$ enters into the basis because it contains a negative $z_2 - c_2$ but $x_6$ exits from the basis. Then, the new basic vector is:

$$
x_1 = 400, \quad x_2 = 50, \quad x_3 = 550, \quad x_4 = 0
$$

In order to compare this result with the results presented in Table 2, we first need to apply the changes in the coefficients and arrange them as:

$$
e_1 = 0.297, \quad e_2 = 0.252 + 0.01 = 0.262, \quad e_3 = 0.268 - 0.01 = 0.258 \text{ and } e_4 = 0.183.
$$

Therefore: $e_1 > e_2 > e_3 > e_4$ which this order is in line with the 4th row of Table 2 and thus the results are the same.

B) Variations of $c_1$ and $c_3$

Step 1: determine the desired value of change in $\Delta c_1 = -0.015$

$\Delta c_1 + \Delta c_3 = 0 \quad \rightarrow \quad \Delta c_3 = 0.015$

Step 3: are the changed parameters related to basic variables?

No

Step 4: do the changed parameters satisfy the 100 percent rule?

$$
\Delta c_1 \geq \frac{1}{\frac{\partial H}{\partial c_1}} - \frac{1}{\frac{\partial H}{\partial c_3}}\left(\frac{\Delta c_1^{\text{max}}}{\Delta c_3^{\text{max}}}\right)
$$

The values of $\Delta c_1^{\text{max}}$ and $\Delta c_3^{\text{max}}$ are extracted from Table 3.

$$
\Delta c_1 \geq \frac{1}{\frac{\partial H}{\partial c_1}} - \frac{1}{\frac{\partial H}{\partial c_3}}\left(\frac{0.015}{0.0145}\right) = -0.0145
$$

$$
-0.0115 < -0.0145
$$

That means the optimality of the basic solution is not kept. If the changes of step 1 and 2 are applied, the new basis may be as follows:

$$
x_B = (x_5, x_3, x_1, x_8, x_7, x_9) = (2600, 700, 500)
$$

The new basis only differs in the auxiliary variable $x_2$ which was replaced by $x_9$ in the previous basis. This means that the values of the decision variables $x_1$ and $x_3$ are not changed. Arranging the updated objective function coefficient values, we have: $c_3 > c_1 > c_2 > c_4$

This order is in line with the 2nd row of Table 2, so the obtained results are the same.
Example 2: Consider the Following LP Problem:

maximize \quad z = x_1 + 2x_2
Subject to:
\begin{align*}
x_1 + x_2 & \leq b_1 \\
-2x_1 + 3x_2 & \leq b_2 \\
2x_1 - 3x_2 & \leq b_3 \\
-3x_1 + 5x_2 & \geq b_4 \\
x_1, x_2 & \geq 0
\end{align*}

Suppose that among the entries of vector b, the following functional relation is defined:

\[ G(b_1, b_2, b_3, b_4) = 0 \]

The initial values of components of b are \( b^T = (8, 9, 6, 15) \). The final (optimal) tableau for this problem is presented in Table 4.

The range for the permitted changes of the RHS parameters is on the basis of the common sensitivity analysis and is presented in Table 5 regardless of the functional relation (Output of Lindo®).

Now assume that \( b_1 \) has changed and \( b_2 \) has to be changed in order to satisfy the functional relation. The following steps are defined in the flowchart shown in Figure 1:

### Table 4. The final tableau for example 2

<table>
<thead>
<tr>
<th>RHS</th>
<th>( x_0 )</th>
<th>( x_5 )</th>
<th>( x_4 )</th>
<th>( x_3 )</th>
<th>( x_2 )</th>
<th>( x_1 )</th>
<th>( z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>0</td>
<td>0</td>
<td>1/5</td>
<td>7/5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>-\frac{1}{5}</td>
<td>\frac{3}{5}</td>
<td>0</td>
<td>1</td>
<td>( x_1 )</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>\frac{1}{5}</td>
<td>\frac{2}{5}</td>
<td>1</td>
<td>0</td>
<td>( x_2 )</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( x_5 )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>\frac{8}{5}</td>
<td>\frac{1}{5}</td>
<td>0</td>
<td>0</td>
<td>( x_6 )</td>
</tr>
</tbody>
</table>

### Table 5. The range for permitted changes in the right-hand side parameters if the functional relation between components of b is ignored

<table>
<thead>
<tr>
<th>RHS</th>
<th>INC</th>
<th>DEC</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_1 = 8 )</td>
<td>( \infty )</td>
<td>5</td>
</tr>
<tr>
<td>( b_2 = 9 )</td>
<td>15</td>
<td>0.625</td>
</tr>
<tr>
<td>( b_3 = 6 )</td>
<td>( \infty )</td>
<td>15</td>
</tr>
<tr>
<td>( b_4 = 15 )</td>
<td>1</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>
Step 1: determine the desired value of change in the first parameter: $\Delta b_1 = -2$

Step 2: calculate the value of change in the second parameter using $G$:

$$7\Delta b_1 + 6\Delta b_2 = 0 \quad \Rightarrow \quad \Delta b_2 = 2\frac{1}{3}$$

Step 3: are the changed parameters related to the binding constraints?

Yes, according to the simplex table, the first and second constraints are binding ones.

Step 4: is the 100 percent rule of changes satisfied?

Yes.

Because of $\Delta b_1 < 0$, the equation (18) is used. Thus, because of the decrease in $b_1$ and the increase in $b_2$, $\Delta b_1^{\text{max}}$ is the maximum value of the permitted decrease of $b_1$, and, $\Delta b_2^{\text{max}}$ is the maximum value of the permitted increase of $b_2$; therefore, $\Delta b_2^{\text{max}} = 15$ and $\Delta b_1^{\text{max}} = -5$ (according to Table 5); and

$$\Delta b_1 \geq \frac{1}{5} \frac{-7/6}{15} \quad \Rightarrow \quad \Delta b_1 \geq -3.6$$

But since $-2 > -3.6$, the desired changes do not make the optimal solution infeasible.

Step 6: calculate the changes of $z^*$ and $x_{b_1}^*$.

According to equations (13) and (14):

$$\Delta z^* = \left[ \frac{7}{5} + \frac{1}{5} \left( -\frac{7}{6} \right) \right] (-2) = -\frac{7}{3}$$

$$\Delta x_1 = \frac{3}{5} - \frac{1}{5} \left( -\frac{7}{6} \right) (-2) = -\frac{5}{3}$$

$$\Delta x_2 = \frac{2}{5} + \frac{1}{5} \left( -\frac{7}{6} \right) (-2) = -\frac{1}{3}$$

The validity of the results is examined by studying the change in the optimal solution through the problem constraints. The first and second constraints identify the optimal solution, so that the changed equations are (see Box1).

According to the simplex table, it is clear that $\Delta x_1 = -\frac{2}{3}$ and $\Delta x_2 = -\frac{1}{3}$ which are in accordance with the obtained result from the flowchart.

Example 3: Consider the LP Problem Presented in Example 2 with the Following Functional Relation

$$G(b_1, b_2, b_3, b_4) = (b_1 - 1)^2 - (b_2 - 2)^2 + (b_3 - 3)^2 + (b_4 - 4)^2 - 2 = 0$$

Suppose $b_1$ has changed and $b_2$ has to be changed in order to satisfy the above functional relation.

Steps 1 and 2: if $\Delta b_1 = 1$, then:

$$dG = 2(b_1 - 1)\Delta b_1 - 2(b_2 - 2)\Delta b_2 = 0$$

$14 - 14\Delta b_2 = 0 \quad \Rightarrow \quad \Delta b_2 = 1$

The changed parameters are related to binding constraints, but the change of $b_1$ is toward unlimited increase, because $b_1$ is able
to be limited when \( \frac{dx_i}{dh} < 0 \) (see equation (15)).

[step 3 & 4], for this reason, it is observed that:

\[
\frac{dx_*}{db_1} = y_{i1} + y_{i2} \left( -\frac{\partial G}{\partial b_1} \right)
\]

\[
\frac{dx_*}{db_2} = \frac{3}{5} - \frac{1}{5} \left( -\frac{14}{14} \right) = \frac{2}{5}
\]

\[
\frac{dx_*}{db_3} = \frac{2}{5} + \frac{1}{5} \left( -\frac{14}{14} \right) = \frac{3}{5}
\]

\[
\frac{dx_*}{db_4} = 0 + 1 \left( -\frac{14}{14} \right) = 1
\]

\[
\frac{dx_*}{db_5} = \frac{1}{5} + \frac{8}{5} \left( -\frac{14}{14} \right) = \frac{9}{5}
\]

Therefore, the basic solution remains feasible in the direction of the determined changes. It is because all the slopes are positive.

Example 4: Consider the Following LP Problem

maximize \( z = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 \)

Subject to:

\[
\begin{align*}
3x_1 + 4x_2 - x_3 & \leq 24 \\
x_1 + 2x_3 - 2x_4 & \leq 36 \\
x_1 + x_2 + x_3 + x_4 & \leq 10 \\
x_1, x_2, x_3, x_4 & \geq 0
\end{align*}
\]

Assume that the functional relation between components of \( c \) is as follows:

\[
G(c_1, c_2, c_3, c_4) = c_1^2 + c_2^2 + 2c_3 - 2c_4 - 25 = 0
\]

We find the optimal table presented in Table 6 assuming an initial value of \( c = (2, -3, 4, -2)^T \) for vector \( c \). Furthermore, Table 7 presents the range of changes in the objective function coefficients (output: LinDo®).

Now suppose that one wants to examine the effect of \( c_1 \) on \( z^* \) and the \( z_j - c_j \) s, so that parameters \( c_2 \) and \( c_3 \) can change (the change of three parameters). The following steps are defined in the flowchart shown in Figure 2.

The desired change of the two parameters is applied in the functional relation and the value of change in the third parameter is obtained.

Step 1: \( \Delta c_1 = \Delta c_2 = \frac{1}{2} \)

Step 2: \( 2\Delta c_1 + 2\Delta c_2 + 2\Delta c_3 = 0 \rightarrow \Delta c_3 = -1 \)

Step 3: parameters are related to basic and non-basic variables.

Step 4: to satisfy relation (36), we have,

\[
\sum_{k=1}^{n} \frac{\partial (z_j - c_j)}{\partial c_k} \frac{dc_k}{dc_k} = \frac{\partial (z_j - c_j)}{\partial c_1} \frac{dc_1}{dc_1} + \frac{\partial (z_j - c_j)}{\partial c_2} \frac{dc_2}{dc_2} + \cdots + \frac{\partial (z_j - c_j)}{\partial c_n} \frac{dc_n}{dc_n}
\]
Table 6. The final tableau for example 4

<table>
<thead>
<tr>
<th>RHS</th>
<th>$x_7$</th>
<th>$x_6$</th>
<th>$x_5$</th>
<th>$x_4$</th>
<th>$x_3$</th>
<th>$x_2$</th>
<th>$x_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>0</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>34</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>16</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>-4</td>
<td>0</td>
<td>-2</td>
<td>-1</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

\[
\frac{d(z_1 - c_i)}{dc_i} = (-1) \times 1 + 0 + 1 \times \frac{\Delta c_i}{\Delta c_1} = -1 + \frac{-1}{1/2} = -3
\]

\[
\frac{d(z_2 - c_i)}{dc_i} = 0 + (-1) \frac{\Delta c_i}{\Delta c_1} + 1 \times \frac{\Delta c_i}{\Delta c_1} = -1 + 1 + \frac{-1}{1/2} = -3
\]

\[
\frac{d(z_4 - c_i)}{dc_1} = 0 + 0 + 4 \times \frac{\Delta c_i}{\Delta c_1} = 4(\frac{-1}{1/2}) = 8,
\]

\[
\frac{d(z_7 - c_i)}{dc_1} = 0 + 0 + 1 \times \frac{\Delta c_i}{\Delta c_1} = 1(\frac{-1}{1/2}) = -2
\]

\[
\Delta c_1 \leq \min \left\{ -\frac{2}{-3}, -\frac{7}{-3}, -\frac{4}{-2} \right\} = \frac{2}{3}
\]

Since \( \frac{1}{2} < \frac{2}{3} \), the basic solution remains optimal.

**Step 5:** by using (35), we have:

\[
\Delta(z_1 - c_i) = -3\Delta c_1 = \frac{-3}{2}, \quad \Delta(z_2 - c_i) = -3\Delta c_1 = \frac{-3}{2}
\]

\[
\Delta(z_4 - c_i) = 8\Delta c_1 = 4, \quad \Delta(z_7 - c_i) = -2\Delta c_1 = -1
\]

Also, by using (34), we have:

\[
\Delta z^* = \left[ 0 + 10(\frac{-1}{1/2}) \right] \Delta c_1 = -10
\]

Table 7. The permitted bound of changes in the objective function coefficients if the functional relation between components of \( c \) is ignored

<table>
<thead>
<tr>
<th>Var.</th>
<th>OFC</th>
<th>INC</th>
<th>DEC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>2</td>
<td>2</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>-3</td>
<td>7</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$x_3$</td>
<td>4</td>
<td>$\infty$</td>
<td>2</td>
</tr>
<tr>
<td>$x_4$</td>
<td>-2</td>
<td>6</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>
CONCLUSION AND FUTURE RESEARCH DIRECTIONS

The importance of sensitivity analysis in linear programming has been widely stressed in the management science literature. However, the research on sensitivity analysis with functional relation has been sporadic and scattered. Usually variation occurs in the RHS of the constraints and/or the objective function coefficients. In this paper, we considered the sensitivity analysis for a class of LP problems with functional relation among the objective function parameters or those of the RHS. The classical methods and standard sensitivity analysis software packages fail to function when a functional relation among the LP parameters prevails. In order to overcome this deficiency, we derived a series of sensitivity analysis formulae and devised corresponding algorithms for different group of homogenous LP parameters. The validity of the derived formulae and devised algorithms was corroborated by open literature examples having linear as well as nonlinear functional relations between their vector b or vector c components.

The derived formulae and the devised procedures developed in this study are not automated. There are no theoretical obstacles to the development and implementation of computer programs for performing sensitivity analysis in LP problems with functional relation. We plan to automate these formulae and procedures. Further research is also needed to develop explicit formulae that describe the behavior of the optimal value under linear and nonlinear perturbations of the constraint coefficient matrix. The research on sensitivity analysis could also benefit from the extension of the current method to multi-dimensional functional relations. Finally, the proposed approach can be extended to cases with probabilistic information on homogenous parameters (e.g., when the parameters are correlated then the change of a specific parameter causes the simultaneous change of others).

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REFERENCES


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