A methodology for analyzing the transient availability and survivability of a system with repairable components

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Abstract

In this paper we present a method for transient analysis of availability and survivability of a system with the identical components and identical repairmen. The considered system is supposed to consist of series of $k$-out-of-$n$ or parallel components. We employed the Markov models, eigen vectors and eigenvalues for analyzing the transient availability and survivability of the system. The method is implemented through an algorithm which is tested in MATLAB programming environment. The new method enjoys a stronger mathematical foundation and more flexibility for analyzing the transient availability and survivability of the system.

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1. Introduction

Reliability has been a major concern for the system designers. Redundancy of components is usually required to design highly reliable systems. A common form of redundancy is $k$-out-of-$n$: $G$ system in which at least $k$-out-of-$n$ components must be good for the system to be good [1]. Consider a system having $n$ identical components. In parallel systems the failure occurs when all of its $n$ components fail. In series systems the failure occurs if at least one of its components fails. Redundancy of components is usually incorporated in a system design for increasing the system reliability.

Many systems consist of components having various failure modes. Several authors have considered a $K$-out-of-$N$ system subject to two failure modes. Among those, Moustafa [2] has presented Markov models for analyzing the transient reliability of $K$-out-of-$N$: $G$ systems subject to two failure modes. He proposed a procedure for obtaining closed form of the transient probabilities and the reliability for non-repairable systems. He then extended his work by providing a set of simultaneous linear differential equations for repairable and non-repairable of two different $N$-out-of-$N$: $G$ systems subject to $M$ failure modes [3]. In his paper

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numerical solutions for the reliability of the repairable systems were discussed, and closed formula for solutions of the reliability for the non-repairable systems were presented. Another research effort is the work of Pham and Pham [4] which has considered \([k, n - k + 1]-\text{out-of-}n: F\) systems subject to two failure modes. Shao and Lamberson presented a model for \(k\)-out-of-\(n\): \(G\) system with load sharing [5].

Zhang et al. [6] presented circular consecutive 2-out-of-\(n\) repairable system with one repairman. They determined rate of occurrence of failure, mean time between failures, reliability, and mean time to first failure. Li et al. [7] presented a \(k\)-out-of-\(n\) system with independent exponential components. They assigned that some working components are suspended as soon as the system is down, repair starts immediately when a component fails and repair times are independent and exponentially distributed. Also they determined mean time between failures, mean working time in a failure-repair cycle, and mean down time in a failure-repair cycle.

Another attempt is the work conducted by Sarhan and Abouammoh [8] who applied the concept of shock model to derive the reliability function of a \(k\)-out-of-\(n\) non-repairable system with non-independent and non-identical components. Later El-Gohary and Sarhan [9] extended Sarhan and Abouammoh work by proposing a Bayes estimator for a three non-independent and non-identical component series system under the condition of four sources of fetal shock. They support their estimation method by presenting a simulation study and showed how one can utilize the theoretical results obtained in their paper.

Azaron et al. [10] introduced a new methodology, by using continuous-time Markov processes and shortest path technique, for the reliability evaluation of an \(L\)-dissimilar-unit non-repairable cold-standby redundant system.

In this paper we present a method for transient analysis of availability and survivability of a system with repairable components using Markov models, eigenvalues and eigen vectors. The considered system is supposed to consist of \(n\) identical components and \(k\) repairmen which components are arranged in series or in \(k\)-out-of-\(n\) or in parallel. We propose a methodology for obtaining availability, survivability, \(MTTFs\) (Mean time to system failure) of the system and calculating the duration for the system to reach to its steady state.

This paper is organized as follows:

We present nomenclature and definitions in Section 2, the model in Section 3, the proposed methodology in Section 4, numerical example in Section 5 and conclusion in Section 6.

2. Nomenclature and definitions

\(N(t)\): number of components failed before time \(t\),
\(N'(t)\): number of repaired components before time \(t\),
\(X(t)\): number of failed components at time \(t\)

\[ X(t) = N(t) - N'(t), \quad (1) \]

\(p_n(t)\): probability of having \(n\) failed components at time \(t\)

\[ p_n(t) = P(X(t) = n), \quad (2) \]

\(A(t)\): probability of system to be up (good) at time \(t\), regardless of its historical components failure and/or repair,

\(A(\infty)\): long time system availability or system reliability,

\(R_s(t)\): survivability function. Determines the probability that a system does not leave the set \(B\) of functioning states during the time interval \((0, t]\)

\[ R_s(t) = \sum_{j \in B} p_j(t), \quad (3) \]

\(MTTFs\): mean time to system failure

\[ MTTF_s = \int_0^{+\infty} R_s(t) \, dt. \quad (4) \]
Definition 1. If we consider $Q$ as the state transient rate matrix and $P(t)$ as the state transient probability in the exponential Markov chain with the continuous time, then we have

1. $P'(t) = P(t) \cdot Q,$
2. $P_n(t) = P_n(0) \cdot P(t).$  

(5)

In which $Q$ and $P(t)$ are square matrixes and, where $P_n(t)$ and $P_n(0)$ are row vectors.

3. The model

In this paper our aim is the determining of availability, survivor function and MTTF of a system with the following assumptions:

1. The system consists of $n$ identical and independent components.
2. The components of system are repairable.
3. The components of system are series or $k$-out-of-$n$ or parallel.
4. The system consists of $k$ identical repairmen.
5. The lifetime of each component is exponentially distributed with the parameter $\lambda$.
6. The service time of each component by each repairman is exponentially distributed with the parameter $\mu$.

4. The proposed methodology

To describe the proposed methodology for analyzing the system’s transient reliability, consider a system having $n$ identical components and $k$ repairmen. The components can be arranged in parallel or series. Our methodology can analyze three distinct cases. In the case of parallel components, we consider that the system fails when all $n$ components fail. In case of series components, we consider that system fails when a component fails. We also consider the case in which $k$-out-of-$n$ components failure. It is assumed that the time between two components failure is a random variable having the exponential distribution with the parameter $\lambda$. We also assume there are $k$ identical repairmen providing services to the system. The service time of a component is also an exponentially distributed random variable with the parameter $\mu$. Our goal is to provide a methodology for analyzing the transient availability and survivability of the system and the time until the system is reached to its steady state. Considering $X(t)$ as the number of failed components at time $t$, we will have the Markov model in Fig. 1.

As an example if we let $n = 4$ and $k = 3$ the Markov model is represented in Fig. 2.

The proposed methodology for obtaining the system availability and the transient probabilities are based on several theorems. These theorems are established to provide the underlying theory of our methodology. We first present these theorems as the following:

![Fig. 1. State transition diagram of the system with $k$ repairmen.](image)
Theorem 1. Let us consider a continuous time exponential Markov chain in which \( P'(t) = P(t) \cdot Q \), then we have
\[
P(t) = e^{Q \cdot t},
\]
\[
P_n(t) = P_n(0) \cdot e^{Q \cdot t}.
\] (6)

Proof
\[
P'(t) = P(t) \cdot Q \Rightarrow \frac{dP(t)}{dt} = P(t) \cdot Q \Rightarrow \frac{dP(t)}{P(t)} = Q \cdot dt,
\]
\[
\int \frac{dP(t)}{P(t)} = \int Q \cdot dt \Rightarrow \ln P(t) = Q \cdot t + C \cdot I \Rightarrow e^{\ln P(t)} = e^{Q \cdot t} \cdot e^{C \cdot t} \Rightarrow P(t) = e^{Q \cdot t} \cdot e^{C \cdot t}.
\]
In which \( I \) is an identity matrix. Since \( P(0) = I \) then we have \( P(t) = e^{Q \cdot t} \). By Definition 1 we will have
\[
P_n(t) = P_n(0) \cdot P(t) = P_n(0) \cdot e^{Q \cdot t}. \]

Consider the following theorem [11].
Let us consider \( Q \) as an \( n \times n \) square matrix which has \( n \) non-repeating eigenvalues, then we have
\[
e^{Q \cdot t} = V \cdot e^{d \cdot t} \cdot V^{-1},
\] (7)
where in which \( t \) represent time, \( V \) is a matrix of eigen vectors of \( Q \), \( V^{-1} \) is the inverse of \( V \) and \( d \) is a diagonal eigenvalues of \( Q \) defined as follows:
\[
d = \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{bmatrix}.
\]
And the matrix \( e^{d \cdot t} \) is as follows:
\[
e^{d \cdot t} = \begin{pmatrix}
e^{\lambda_1 \cdot t} & 0 & \cdots & 0 \\
0 & e^{\lambda_2 \cdot t} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & e^{\lambda_n \cdot t}
\end{pmatrix}.
\]

Theorem 2. Consider \( P(t) = e^{Q \cdot t} \) in which \( Q \) is the transition matrix. In matrix \( Q \) one of the eigenvalues is zero and the remaining eigenvalues are the complex number with the negative real part.

Proof. Since in every row of transition matrix the summation of row elements is zero, we can deduce that one its eigenvalue of matrix \( Q \) is zero. By Theorem 1 and relation (7), we have
\[
P(t) = V \cdot e^{d \cdot t} \cdot V^{-1} = (p_{ij}(t)),
\]
\[
p_{ij}(t) = \pi_j + \sum_{k=1}^{n-1} \alpha_{ijk} \cdot e^{\lambda_k \cdot t}.
\] (8)
In which $\lambda_k$ is the $k$th eigenvalue, $a_{ijk}$'s are constant values, and $\pi_j$ is the limiting probability. Using the contradictory concept, if we assume that one of the eigenvalues of $Q$ is a complex number with positive real part then we have

$$\lim_{t\to \infty} p_{ij}(t) = \infty,$$

which contradicts $\lim_{t\to \infty} p_{ij}(t) = \pi_j$ and therefore the eigenvalues of $Q$ are complex numbers with the negative real part. □

**Theorem 3.** Consider $P(t) = e^{Qt}$ in which $Q$ is the transition matrix, the time elapse until system reaches to the steady state ($P(t) = \Pi$) can be calculated by the following formula:

$$t = \frac{\ln \varepsilon}{S_r}.$$  \hfill (9)

In which $\varepsilon$ is a very small number (i.e. $\varepsilon = 0.0001$), $S_r$ is the largest real part of the eigenvalues excluding the zero element of matrix $Q$ and $\Pi$ is a square matrix representing the limiting probabilities. The elements of matrix $P(t)$ and $\Pi$ are shown as follows:

$$P(t) = \begin{bmatrix} p_{00}(t) & p_{01}(t) & \cdots & p_{0n}(t) \\ p_{10}(t) & p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n0}(t) & p_{n1}(t) & \cdots & p_{nn}(t) \end{bmatrix},$$

$$\Pi = \begin{bmatrix} \pi_0 & \pi_1 & \cdots & \pi_n \\ \pi_0 & \pi_1 & \cdots & \pi_n \\ \vdots & \vdots & \ddots & \vdots \\ \pi_0 & \pi_1 & \cdots & \pi_n \end{bmatrix}.$$

**Proof**

$$p_{kj}(t) = \pi_j + \sum_{m=1}^{n-1} a_{kjm} \cdot e^{S_m t} = \pi_j + \sum_{m=1}^{n-1} a_{kjm} \cdot e^{(S_m + C_m) t}.$$  

By Theorem 2 all $S_m$ are negative, and $i = \sqrt{-1}$ ($\pi_j$, $a_{kjm}$, $S_m$, and $C_m$ are constant numbers). Now suppose $S_r$ is greater then $S_m$, then for large values of $t$ we have

$$\varepsilon = e^{S_r t},$$

$$S_r \cdot t = \ln \varepsilon,$$

$$Yt = \frac{\ln \varepsilon}{S_r}.$$  

Based on the proof of these theorems, we now propose an algorithmic procedure for calculating the availability and survivability of the system. □

**Algorithm**

1. Let $i = 0$.
2. Determine the transition matrix $Q$.
3. Determine the eigenvalues and eigen vectors of the matrix $Q$ and Let $i = i + 1$.
4. Determine $P(t) = V \cdot e^{\lambda t} \cdot V^{-1}$.
5. Determine $P(t) = P_n(0) \cdot P(t)$ and if $i = 1$ go to step 6 and if $i = 2$ go to step 7.
6. Determine the availability of the system according to the type of the system as follows:

- For a system with parallel components
  \[ A(t) = 1 - p_n(t). \]  
  \( \text{(10)} \)

- For a system with series components
  \[ A(t) = p_0(t). \]  
  \( \text{(11)} \)

- For \( k \)-out-of-\( n \) system
  \[ A(t) = \frac{\sum_{j=0}^{n-k} p_j(t)}{n}. \]  
  \( \text{(12)} \)

then delete the \( n \)th row and \( n \)th column of the matrix \( Q \) and go to step 3.

7. Determine the survivability and MTTFs of the system with parallel components as follows:

\[ R_s(t) = \sum_{j=0}^{n-1} p_j(t), \quad \text{MTTF}_s = \int_0^{+\infty} R_s(t) \, dt. \]  
(13)

5. A numerical example

Consider a system having five identical components. There are two identical repairmen for repairing this system. It is assumed that time to failure of repaired component is a random variable with exponential distribution function with the mean of 1/2 h. The repair time is also considered to be a random variable distributed exponentially with the mean of 1/10 of hour. We would like to calculate availability, survivability of the system at any given time, considering the following system configurations:

1. Components are arranged in parallel.
2. Components are arranged in series.
3. System is working if at least 2-out-of-5 is good.

\textit{Solution.}

For determining of the system availability we have

The graphical Markov model can be represented as in Fig. 3.

According to the algorithm we have

\[ Q = \begin{bmatrix}
-10 & 10 & 0 & 0 & 0 & 0 \\
10 & -18 & 8 & 0 & 0 & 0 \\
0 & 20 & -26 & 6 & 0 & 0 \\
0 & 0 & 20 & -24 & 4 & 0 \\
0 & 0 & 0 & 20 & -22 & 2 \\
0 & 0 & 0 & 0 & 20 & -20 \\
\end{bmatrix}, \]

\[ \text{Fig. 3. State transition diagram of the system with 2 repairmen.} \]
Now we can calculate the system availability for different type of the system configurations as follows:

1. Components are arranged in parallel

\[
A(t) = 1 - p_5(t),
\]

\[
A(\infty) = 0.999. \tag{15}
\]

2. Components are arranged in series

\[
A(t) = p_0(t),
\]

\[
A(\infty) = 0.392. \tag{16}
\]

3. The system is working if 2-out-of-5 is good

\[
A(t) = p_3(t) + p_2(t) + p_1(t) + p_0(t) = 1 - p_4(t) - p_5(t),
\]

\[
A(\infty) = 0.9897. \tag{17}
\]

We can calculate the time elapse until system reaches to the steady state. Table 1 represents the probability of the system to be up (good) at time \( t \), for different values of \( t \) (probated by 0.05 unit) and according to the different system configurations.

By the following closed form formula, we can also calculate the time elapse until system reaches to the steady state.

\[
t = \frac{\ln e}{S_r} = \frac{\ln 0.0001}{-9.02} = 1.
\]

As it can be seen from both Table 1 and the closed form formula, the system reaches to the steady state after one unit time.

The limiting probability can also calculated as follows:

\[
\pi_0 = 0.3927, \quad \pi_1 = 0.3927, \quad \pi_2 = 0.1571, \quad \pi_3 = 0.0471, \quad \pi_4 = 0.0094, \quad \pi_5 = 0.0009. \tag{18}
\]

<table>
<thead>
<tr>
<th>( t )</th>
<th>( A(t) ) for parallel</th>
<th>( A(t) ) for series</th>
<th>( A(t) ) for 2-out-of-5</th>
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<td>0.05</td>
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<td>0.9998</td>
</tr>
<tr>
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<tr>
<td>1.50</td>
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<td>0.3927</td>
<td>0.9897</td>
</tr>
</tbody>
</table>
For determining of the system survivability and MTTFs, according to the algorithm we have

\[
Q = \begin{bmatrix}
-10 & 10 & 0 & 0 & 0 \\
10 & -18 & 8 & 0 & 0 \\
0 & 20 & -26 & 6 & 0 \\
0 & 0 & 20 & -24 & 4 \\
0 & 0 & 0 & 20 & -22
\end{bmatrix},
\]

\[
P_0(t) = 0.0166e^{-42.1t} + 0.396e^{-0.0168t} + 0.0668e^{-29.4t} + 0.270e^{-9.94t} + 0.251e^{-18.5t},
\]

\[
P_1(t) = -0.0534e^{-42.1t} + 0.395e^{-0.168t} - 0.130e^{-29.4t} + 0.00163e^{-9.94t} - 0.214e^{-18.5t},
\]

\[
P_2(t) = 0.056e^{-42.1t} + 0.157e^{-0.0168t} + 0.0405e^{-29.4t} - 0.134e^{-9.94t} - 0.12e^{-18.5t},
\]

\[
P_3(t) = -0.0238e^{-42.1t} + 0.0464e^{-0.0168t} - 0.045e^{-29.4t} - 0.108e^{-9.94t} + 0.0408e^{-18.5t},
\]

\[
P_4(t) = 0.00473e^{-42.1t} + 0.00845e^{-0.0168t} - 0.0243e^{-29.4t} - 0.036e^{-9.94t} + 0.0471e^{-18.5t}.
\]

Now we can calculate the system survivability and MTTFs for parallel components as follows:

\[
R_s(t) = \sum_{j=0}^{4} p_j(t), \quad \text{MTTF}_s = 59.5999.
\]

6. Conclusion

In this paper we proposed a methodology for analyzing system transient survivability and availability with identical components and identical repairmen. We employed the Markov models, eigen vectors and eigenvalues concepts to develop the methodology for the transient reliability of such systems. The proposed methodology is a more effective method in the sense that it can be applied for analyzing the varieties of systems, i.e. series, parallel and $k$-out-of-$n$ systems. The proposed methodology can also be employed for determining MTTFs and the time elapse until system reaches to the steady state.

References